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Sensitivities of multiple singular values for optimal geometries of precision structures

S. Hakim, M. B. Fuchs*

The Iby and Aladar Fleischman Faculty of Engineering, Tel Aviv University, 69978 Ramat Aviv, Israel

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Abstract

This paper presents the sensitivities of a repeated singular value of a matrix with respect to perturbations in that matrix. The difficulty in computing the sensitivities of a repeated singular value is linked to the fact that the multiplicity of the singular value may change during the perturbation. The derivative is developed based on an approach used for repeated eigenvalues of self adjoint systems, by constraining the singular values to remain bundled during the perturbation. The need for the sensitivities of singular values arose when optimizing the geometry of precision structures under a family of disturbances characterized by a disturbance influence matrix. The aim was to modify the geometry of the structure in a way which enhances its performance. Since the structure is subjected to a multitude of loading cases the objective is to minimize the worst possible distortion. It is shown that this is equivalent to minimizing the first singular value of the disturbance influence matrix. Consequently, in the mathematical programming formulation the objective function is the first singular value under the constraints inherent to the method for computing the sensitivities. This is then solved by a Lagrangian method. It is shown that the technique is very reliable as visualized in two typical truss examples. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Precision structures support devices requiring a high degree of accuracy such as large antennas or telescope mirrors. The purpose of the supporting part of the structure is to maintain small distortions under a family of external disturbances. In this paper the precision structures are elastic trusses and the distortions are the displacements at some of the nodes. The degrees of freedom whose displacements have to be kept to a minimum are called the controlled degrees of freedom (CDOF). The disturbance is represented by an influence matrix, $\boldsymbol{D}$, where a column, i of $\boldsymbol{D}$ is the distortion obtained by applying a unit disturbance at the ith disturbance source. The task is to design the geometry of a truss such as to maintain the CDOF as undeformed as possible under all

* Corresponding author.

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combinations of disturbances. Since the structure is subjected to many loading conditions the objective function was selected as the worst possible distortion under any combination of disturbances. It is shown that this is equivalent to minimize the first singular value of the disturbance influence matrix (Hakim and Fuchs, 1996). Minimizing the singular value with respect to nodal coordinates lead to the need for determining the sensitivities of a singular value when the matrix is a function of a set of variable parameters.

The central topic of this paper is the computation of the gradient of a singular value of a matrix and in particular in the case of repeated singular values. Note, the matrix need not be square. The derivation of the gradient of distinct singular values is straightforward (Junkins and Kim, 1990). In the repeated singular value case it is developed following the method of Masur (1984, 1985) for the derivative of repeated eigenvalues of self adjoint systems.

The optimization procedure, adapted from Czyz and Lukasiewicz (1995), where it is used for eigenvalues of symmetric matrices, is based on a constrained optimization with a Lagrange multipliers matrix. The expected maximum multiplicity of the singular value is determined in advance and updated iteratively. During the optimization the algorithm checks whether repeated singular values occurred in order to determine the next step in the design space. The algorithm stops when there is no further improvement.

The sensitivities of the singular value depends on the derivative of the disturbance influence matrix with respect to geometry variations, $\partial \boldsymbol{D}/\partial x_i$, where x_i is the coordinate of one of the nodes. The disturbance influence matrix, $\boldsymbol{D}$, is in fact the response of part of the structure to external excitation. Therefore the derivative we seek is that of a subset of the displacement vector under specified loadings, with respect to geometrical variations. For truss structures that derivative may be obtained analytically, using standard sensitivity analysis, as in Adelman and Haftka (1986), combined with an adaptation of the methods used for large displacements, Levy and Spillers (1995). The 2-D case will be presented herein. A column of $\boldsymbol{D}$ is the displacement at the CDOF due to one of the disturbance sources, hence, the derivative of that displacement will be the appropriate column of $\partial \boldsymbol{D}/\partial x_i$.

The optimization algorithm was implemented in Matlab (Moler et al., 1987) and tested on a two hinged bridge for two cases. In the first case we had a distinct singular value, whereas in the second, a repeated singular value occurred. The numerical implementation proved to be very reliable. In the second example the proposed optimization algorithm detected the repeated singular values in due time and reduced them simultaneously without difficulty. A counterexample emphasized the kind of difficulties one can expect when ignoring the multiplicity of the singular values. From a conceptual viewpoint, the results show that significant improvements are achieved by using optimal geometries.

In the next section we formulate the problem for designing optimal precision structures under a family of disturbances. Using the nodal coordinates as design variables one seeks to minimize the first singular value of the disturbance influence matrix while keeping the volume of material constant. In Section 3 we develop the sensitivity relations of a singular value of a matrix with respect to parameters affecting the matrix. It is shown that in the general case of a repeated singular value there exists an implicit relation between a variation of the parameter and a variation of the singular value. We point out in the following section that by constraining the parameters to a subspace which maintains the equality between the singular values of the multiplicity group, explicit sensitivities (derivatives) can be computed. Consequently the minimization formulation in

Section 2 is augmented by equality equations constraining the parameter space to maintain the multiplicity of the singular value. It is indicated that the constrained problem is solved by a Langrangian multipliers method. Section 5 deals with computing the sensitivities of a singular value in the particular case of a disturbance influence matrix with respect to nodal coordinates. Numerical examples are presented in Section 6 and the paper terminates with conclusions in Section 7.

2. Designing precision structures

The need for shape precision arises when the structure has to support devices requiring a high degree of accuracy, such as antenna or multi-lens telescopes. Consider for instance the bridge in Fig. 1, having 40 DOF and 51 axial elements. Assume that the precision requirements are on the lateral displacements of the lower chord. These nine displacements are the CDOF. Due to some external agent the truss will experience nonzero displacements, in particular at these CDOF. The design aim is to reduce the magnitude of the displacements at the CDOF. In the following we will term the external agents which affect the structure disturbances, and the resulting displacements at the CDOF distortions. Examples of disturbances are temperature gradients, element size errors, moving loads and so forth.

In this work we deal with passive control, as opposed to active control where sensors and actuators are embedded in the structure (Hakim and Fuchs, 1996). Passive design relies on the structure itself for reducing the distortion. We will assume that there are N_d disturbance sources and N_p CDOF. These distortions can be represented by

$$\boldsymbol{v}_d = \boldsymbol{D}\boldsymbol{d} \tag{1}$$

where $\boldsymbol{v}_d$ is the N_p displacements at the CDOF due to the disturbance, $\boldsymbol{D}$ is the $N_p \times N_d$ disturbance influence matrix and $\boldsymbol{d}$ is a disturbance vector of size N_d. Matrix $\boldsymbol{D}$ is assumed to be known; however, the disturbances $\boldsymbol{d}$ are arbitrary. Note that $\boldsymbol{v}_d$ is the displacements of the CDOF and not of all the nodes. For precision control it is customary to minimize the RMS (Root Mean Square) of the distortion, that is to minimize $\boldsymbol{v}_d^T\boldsymbol{v}_d$. This demand can be formulated as a performance measure

$$J \equiv \boldsymbol{v}_d^T\boldsymbol{v}_d = \boldsymbol{d}^T\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{d} \tag{2}$$

The value of J can be obtained for every disturbance. Our interest lies in the largest possible value of J, denoted J^*. This is the worst case error, no other disturbance will yield larger RMS of the distortion.

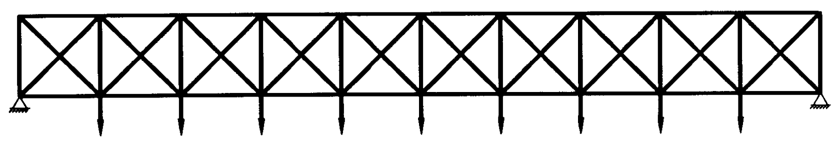

Fig. 1. Bridge truss with lower-chord CDOF.

To compute J^* we assume that the disturbance space $\boldsymbol{D}$ is such that the disturbances are limited by a hypersphere constraint

$$\boldsymbol{d}^T\boldsymbol{d} - 1 = 0 \tag{3}$$

where the radius is unitary without loss of generality. A different radius can be accommodated by a scaling of $\boldsymbol{D}$. The next step is to perform a Singular Value Decomposition (SVD) of $\boldsymbol{D}$ (Golub and Van Loan, 1989)

$$\boldsymbol{U}\boldsymbol{\Sigma}_d\boldsymbol{V}^T = \boldsymbol{D} \tag{4}$$

where $\boldsymbol{\Sigma}_d$ is an $N_p \times N_d$ quasi-diagonal matrix having σ_i, the singular values of $\boldsymbol{D}$, in descending order on the main diagonal, and $\boldsymbol{U}(N_p \times N_p)$, $\boldsymbol{V}(N_d \times N_d)$ are orthonormal square matrices. If $\boldsymbol{D}$ is of full rank then all the diagonal entries of $\boldsymbol{\Sigma}_d$ are greater than zero. Else, if $\boldsymbol{D}$ has say rank r, only the first r σ_i's are different from zero, the remaining ones being null.

Employing (4), eqn (1) can be rewritten as

$$\boldsymbol{v}_d = \boldsymbol{u}_1\sigma_1\tilde{d}_1 + \boldsymbol{u}_2\sigma_2\tilde{d}_2 + \cdots + \boldsymbol{u}_r\sigma_r\tilde{d}_r \tag{5}$$

where $\boldsymbol{u}_i$ is the ith column of $\boldsymbol{U}$ and $\tilde{d}_i$ is the ith component of $\tilde{\boldsymbol{d}} = \boldsymbol{V}^T\boldsymbol{d}$. Recalling that $\boldsymbol{d}^T\boldsymbol{d} = 1$ and that the $\boldsymbol{u}_i$'s are orthonormal, it is clear that the largest size of $\boldsymbol{v}_d$ is σ_1.

Therefore, we propose to use the largest singular value of the disturbance influence matrix as the measure of the precision performance of the structure. Assuming that two structures are suggested for the same family of disturbances, the one with the smaller measure is preferred.

The objective is to find, by moving the nodes of the truss, an optimal geometry with the least value of J^*. The structural elements are assumed to be of constant cross-sections and a constant volume constraint is imposed. Formally the design problem is to find new coordinates of the nodes $\boldsymbol{x}$, while keeping the volume constant, such as to minimize J^*. Since we have shown that σ_1 is a measure of J^* the problem becomes

$$\min_{\boldsymbol{x}} \quad \sigma_1 \quad \textit{subject to} \sum_{i=1}^{M} l_i = \text{constant} \tag{6}$$

where l_i is the length of element i and M is the number of elements in the truss. Please note, depending on the design parameters, $\boldsymbol{x}$ is either the coordinates of all the nodes of the truss or a subset of these coordinates. For example, one could impose that the CDOF locations are constrained while the backup structure may assume any geometry. When solving this mathematical programming problem by a Lagrangian multipliers method the need arises for computing the gradient of σ_1. This is a central topic of this paper and it is addressed in the following two sections.

3. Variations of repeated singular values

In this section we will discuss the sensitivities of singular values of a matrix, and in particular repeated singular values, with respect to parameters which affect the matrix. Derivatives of distinct singular values are given by Junkins and Kim (1990). Let

$$A = U\Sigma V^T \tag{7}$$

be the singular decomposition of a matrix A, then singular value m of A can be expressed as

$$\sigma_m = u_m^T A v_m \tag{8}$$

where u_m and v_m are columns m of U and V, respectively. The variation of the mth singular value with respect to variations in parameters of A is, neglecting higher order terms,

$$\delta\sigma_m = u_m^T \delta A v_m \tag{9}$$

or, if matrix $A(p)$ is a differentiable function of a set of parameters p_k

$$\sum_k \frac{\partial \sigma_m}{\partial p_k}\delta p_k = u_m^T\left(\sum_k \frac{\partial A}{\partial p_k}\delta p_k\right)v_m \tag{10}$$

and also

$$\sum_k \frac{\partial \sigma_m}{\partial p_k}\delta p_k = \sum_k\left(u_m^T \frac{\partial A}{\partial p_k} v_m\right)\delta p_k \tag{11}$$

Since the δp_k are arbitrary, this equality implies that the derivative of a singular value with respect to a parameter p_k is

$$\frac{\partial \sigma_m}{\partial p_k} = u_m^T \frac{\partial A}{\partial p_k} v_m \tag{12}$$

It is also assumed that matrix A is real. If A is complex the derivative is the real part of the r.h.s. of (12). This relation is valid for a distinct singular value.

We intend to utilize these derivatives for reducing the singular value in order to optimize the performance of the structure. In the process of reducing a given singular value it often occurs that smaller singular values increase which, as pointed out by Masur (1984), may result in a crossover. Consequently we have to consider the case of repeated singular values. Repeated singular values are r consecutive singular values having the same value, where r is the multiplicity of the singular value. To obtain their derivative we modify a technique originally developed for repeated eigenvalues of symmetric matrices. Derivatives of repeated eigenvalues for self adjoint systems were presented by Masur (1984, 1985) and for symmetric matrices by Czyz and Lukasiewicz (1995).

Let B be a symmetric matrix, which depends on parameters p, having an eigenvalue λ_0 of multiplicity r. Let M be an $r \times r$ matrix of components

$$m_{ij} = v_i^T \delta B v_j, \quad i,j = 1,\ldots,r \tag{13}$$

where δB is the change in B due to δp and v_i is the eigenvector associated with the ith repeated eigenvalue. It can be shown that the r eigenvalues of M are the variations $\delta\lambda_i$ of eigenvalue λ_0. The r eigenvalues of $B+\delta B$ are now $\lambda_i = \lambda_0 + \delta\lambda_i$.

To extend these results to singular values we recall that singular values of a matrix A are the square roots of the eigenvalues $A^T A$ (Golub and Van Loan, 1989). Denoting $B = A^T A$, then $\lambda_i = \sigma_i^2$, where λ_i is an eigenvalue of B, σ_i is a singular value of A, and the eigenvectors of B are the right singular vectors of A, denoted V in (7).

The relations between variations in $\boldsymbol{B}$ and $\boldsymbol{A}$ are

$$\delta \boldsymbol{B} = \delta \boldsymbol{A}^T \boldsymbol{A} + \boldsymbol{A}^T \delta \boldsymbol{A} \tag{14}$$

and the corresponding relation between variations of the eigenvalues and singular values is

$$\delta \lambda_i = 2\sigma_i \delta \sigma_i \tag{15}$$

Substituting (14) in (13) and using (7) we have

$$m_{ij} = \boldsymbol{v}_i^T \delta \boldsymbol{A}^T (\boldsymbol{U\Sigma V}^T)\boldsymbol{v}_j + \boldsymbol{v}_i^T (\boldsymbol{V\Sigma}^T \boldsymbol{U}^T)\delta \boldsymbol{A}\boldsymbol{v}_j \tag{16}$$

From the orthonormality of $\boldsymbol{V}$ it follows that $\boldsymbol{V}^T\boldsymbol{v}_j$ is a zero vector but for entry j which equals one. Hence, the above relation reduces to

$$m_{ij} = \boldsymbol{v}_i^T \delta \boldsymbol{A}^T \sigma_j \boldsymbol{u}_j + \boldsymbol{u}_i^T \sigma_i \delta \boldsymbol{A} \boldsymbol{v}_j \tag{17}$$

The eigenvalues of $\boldsymbol{M}$ are now the changes in the squares of the singular values. To find the variation of the singular values we use (15) and defining $\sigma_0 = \sigma_i = \sigma_j$ we obtain

$$\delta \sigma = \frac{1}{2\sigma_0} \operatorname{eig}(\boldsymbol{M}) = \operatorname{eig}(\hat{\boldsymbol{M}}) \tag{18}$$

where eig(. . .) are the eigenvalues of the matrix within parenthesis. Finally we obtain matrix $\hat{\boldsymbol{M}}$ of components

$$\hat{m}_{ij} = \tfrac{1}{2}(\boldsymbol{u}_j^T \delta \boldsymbol{A} \boldsymbol{v}_i + \boldsymbol{u}_i^T \delta \boldsymbol{A} \boldsymbol{v}_j) \tag{19}$$

An eigenvalue of $\hat{\boldsymbol{M}}$, $\delta\sigma_i$, is the change in the ith repeated singular value of $\boldsymbol{A}$ due to a variation $\delta\boldsymbol{p}$. The variations of the repeated singular values are an implicit function of the variations of $\boldsymbol{p}$. What we are missing is an explicit relation of the type in (9) which led to the derivative in (12).

4. Sensitivities of repeated singular values

To obtain explicit derivatives, for the optimization algorithm, a further assumption is made based on the following considerations (Masur, 1984). From a numerical point of view the repeated singular values are not strictly equal but packed very closely. In fact one can order them by size. Each one of the singular values has its own derivative. Consequently, for an arbitrary step $\delta\boldsymbol{p}$ in the parameter space some singular values of the multiplicity group may increase while others may decrease. Hence it is reasonable to assume that the order of the singular values will change. This is very detrimental to the efficiency of the minimization. Since cross-overs are likely to occur when we try to reduce the largest singular value, we will have to keep switching the objective function, moving from one singular value to its neighbor.

A common cure to that predicament, presented in Masur (1984), is to keep all the repeating singular values clustered and to reduce them simultaneously. Parameter variations $\delta\boldsymbol{p}$ will therefore be confined to a subspace, $\Delta\bar{\boldsymbol{p}}$, which causes the eigenvalues of $\hat{\boldsymbol{M}}$ to remain equal, that is, $\delta\sigma_i = \delta\sigma_0$ for $i = 1, \ldots, r$. This can only occur if $\hat{\boldsymbol{M}}$ is the diagonal matrix $\hat{\boldsymbol{M}} = \delta\sigma_0\boldsymbol{I}$. Parameter variations, $\delta\bar{\boldsymbol{p}}$, belonging to the subspace $\Delta\bar{\boldsymbol{p}}$ must, therefore, satisfy the conditions $\hat{m}_{ij} = 0$ for $i \neq j$ and $\hat{m}_{ij} = \delta\sigma_0$ for $i = j$. Noting that

$$\delta \boldsymbol{A} = \sum_k \frac{\partial \boldsymbol{A}}{\partial p_k} \delta \bar{p}_k \tag{20}$$

these conditions become with (19)

$$\frac{1}{2}\sum_k \left(\boldsymbol{u}_i^T \frac{\partial \boldsymbol{A}}{\partial p_k} \boldsymbol{v}_j + \boldsymbol{u}_j^T \frac{\partial \boldsymbol{A}}{\partial p_k} \boldsymbol{v}_i \right) \delta \bar{p}_k = 0, \quad i \neq j, \quad i,j = 1, \ldots, r \tag{21}$$

and

$$\boldsymbol{u}_i^T \left(\sum_k \frac{\partial \boldsymbol{A}}{\partial p_k} \bar{\delta} p_k \right) \boldsymbol{v}_i = \delta \sigma_0, \quad i = 1, \ldots, r \tag{22}$$

In subspace $\Delta \bar{\boldsymbol{p}}$ the variation of the singular value can now be expressed as

$$\delta \sigma_0 = \sum_k \left(\boldsymbol{u}_1^T \frac{\partial \boldsymbol{A}}{\partial p_k} \boldsymbol{v}_1 \right) \delta \bar{p}_k \tag{23}$$

Please note, since the singular value increments are the same for all the multiplicities, the indices of $\boldsymbol{u}$ and $\boldsymbol{v}$ are arbitrarily taken as one. Following procedure (10)–(12) we find the common derivative to all the multiple singular values

$$\frac{\partial \sigma_0}{\partial \bar{p}_k} = \boldsymbol{u}_m^T \frac{\partial \boldsymbol{A}}{\partial \bar{p}_k} \boldsymbol{v}_m \tag{24}$$

These are the sensitivities we were looking for. These sensitivities can be computed in a parameter space satisfying (21) and (22).

In practice the design moves in the entire variable space. Following the optimization algorithm, proposed in Czyz and Lukasiewicz (1995), a Lagrangian multipliers matrix is used to impose the constraints (21), (22) and the constant volume constraint in (6). Since the multiplicity of the singular values is not known in advance, and can change during the optimization, a criterion for detecting the modality of the problem is required. The criterion is based on γ. Prior to the optimization the maximum expected modality of the problem, m, is determined. Matrix γ is then calculated and if it is positive definite, m is the modality of the problem, else, m is reduced by 1 and that procedure is continued until $m = 1$. Once the multiplicity is found the step size and direction are determined using the Lagrange multipliers matrix γ. The algorithm is repeated until there is no further improvement, that is, until $\delta\sigma_0 = 0$. Note, to determine the positive definiteness of γ, Czyz and Lukasiewicz (1995) have checked whether the diagonal is positive. This does not seem to suffice since a positive diagonal is a necessary, but not a sufficient, condition for positive definiteness of a matrix.

5. Geometrical derivative of the disturbance influence matrix

In the previous sections we found the derivative of a single or multiple singular value given the derivative of the matrix. In this section we apply the method to computing the derivative of the disturbance influence matrix with respect to the nodal coordinates of a truss. Recall that every column of $\boldsymbol{D}$ is part of a displacement vector, generated from unit disturbance. Therefore, the derivative is in fact a structural response derivative. In the case of truss structures this derivative can be described explicitly.

In order to obtain the derivative of the displacements of a truss structure we start from the structural field equations. Consider a linear elastic truss composed of M members and N nodal degrees of freedom. The equations governing the response of the structure are (1) the elongation–displacement equations or geometric relations, (2) the constitutive relations or generalized Hooke's law and (3) the equilibrium equations

$$\boldsymbol{e} = \boldsymbol{Ru} \tag{25}$$

$$\boldsymbol{t} = \boldsymbol{Se} \tag{26}$$

$$\boldsymbol{f} = \boldsymbol{Qt} \tag{27}$$

where $\boldsymbol{e}$ is the M-vector of geometric elongations of the elements, $\boldsymbol{R}$ is the $M \times N$ kinematics matrix, $\boldsymbol{u}$ is the N-vector of nodal displacements, $\boldsymbol{t}$ is the M-vector of element forces, $\boldsymbol{S}$ is the $M \times M$ diagonal matrix of element stiffness ($S_{jj} = E_j A_j / L_j$), E_j, A_j, L_j are, respectively, Young's modulus, the cross-sectional area and the length of bar j, $\boldsymbol{f}$ is the N-vector of applied nodal loads, and $\boldsymbol{Q} = \boldsymbol{R}^T$ is the $N \times M$ statics matrix.

Using (25)–(27) and denoting $\boldsymbol{K} \equiv \boldsymbol{QSR}$ we have

$$\boldsymbol{Ku} = \boldsymbol{f} \tag{28}$$

Equation (28) can now be solved to find the displacements.

The general displacement derivative with respect to a parameter p, assuming differentiability of all the appropriate matrices, is (Adelman and Haftka, 1986),

$$\boldsymbol{K}\frac{\partial \boldsymbol{u}}{\partial p} = \frac{\partial \boldsymbol{f}}{\partial p} - \frac{\partial \boldsymbol{K}}{\partial p}\boldsymbol{u} \tag{29}$$

After determining the 'pseudo-loads' in the r.h.s. of (29) the sensitivities of the displacements are obtained by solving (29). In many applications, as it is here, the applied forces are constant and therefore $\partial \boldsymbol{f}/\partial p$ is zero. There are, however, cases for which this derivative should be evaluated, for example, if the disturbance forces act along the truss members such as induced thermal stresses. With regard to the sensitivity of the stiffness matrix

$$\frac{\partial \boldsymbol{K}}{\partial p} = \frac{\partial}{\partial p}(\boldsymbol{QSR}) \tag{30}$$

we notice that the matrices whose derivatives must be established are $\boldsymbol{S}$, and $\boldsymbol{Q}$ (or $\boldsymbol{R} = \boldsymbol{Q}^T$). Since we restrict ourselves to geometrical optimization the design parameters are node coordinates denoted $\boldsymbol{x}$. The derivatives of $\boldsymbol{Q}$ and $\boldsymbol{S}$ may be obtained using statistical analysis for large displacements, e.g. Levy and Spillers (1995), which is also concerned with the change in the stiffness matrix due to nodal coordinate changes. In the following we will give the explicit derivatives for 2-D truss structures.

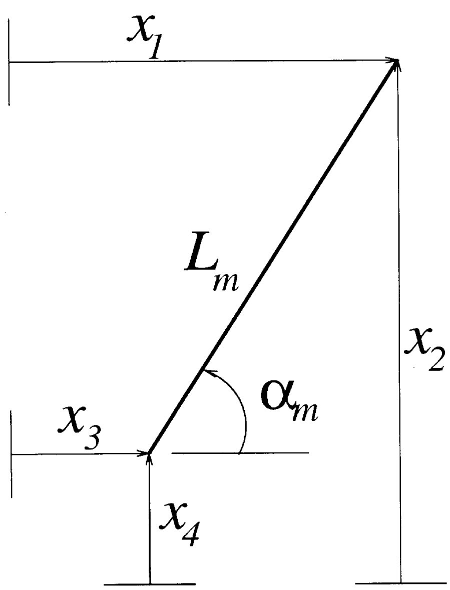

Fig. 2. Typical 2-D bar.

In Fig. 2 we have sketched a typical planar truss element of length L_m making a slope α_m with the horizontal axis. This element corresponds to $\boldsymbol{q}_m$ (column m of $\boldsymbol{Q}$) and to the diagonal component s_m of $\boldsymbol{S}$. We are seeking the sensitivities of $\boldsymbol{Q}$ and $\boldsymbol{S}$ relative to an arbitrary coordinate x_i. In fact we need $\partial s_m/\partial x_i$ and $\partial \boldsymbol{q}_m/\partial x_i$. A variation $\mathrm{d}x_i$ causes changes $\mathrm{d}L_m$ of the length and $\mathrm{d}\alpha_m$ of the slope of all elements connected to that coordinate. Now, s_m is an explicit function of L_m and likewise $\boldsymbol{q}_m$ is an explicit function of α_m only. This suggests the use of the following chain derivation

$$\frac{\partial s_m}{\partial x_i} = \frac{\partial s_m}{\partial L_m}\frac{\partial L_m}{\partial x_i} \tag{31}$$

$$\frac{\partial \boldsymbol{q}_m}{\partial x_i} = \frac{\mathrm{d}\boldsymbol{q}_m}{\mathrm{d}\alpha_m}\frac{\partial \alpha_m}{\partial x_i} \tag{32}$$

We have $s_m = (EA)_m/L_m$ and consequently

$$\frac{\partial s_m}{\partial L_m} = -\frac{s_m}{L_m} \tag{33}$$

We now focus on $\partial L_m/\partial x_i$. Let $\boldsymbol{x}$ and $\boldsymbol{l}$ be, respectively, the N-vector of nodal coordinates and the M-vector of elemental lengths. We modify all the coordinates by a small amount $\mathrm{d}\boldsymbol{x}$. The element lengths vary accordingly by $\mathrm{d}\boldsymbol{l}$. It is rather obvious that the relationship between $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{l}$ (infinitesimals) is conforming to the relationship between $\boldsymbol{u}$ and $\boldsymbol{e}$ (small displacements). Once one accepts the truth of this rule, (25) applies equally to small modifications of the positions of the joints and ensuing element length changes

$$\mathrm{d}\boldsymbol{l} = \boldsymbol{Q}^T \,\mathrm{d}\boldsymbol{x} \tag{34}$$

In other words

$$\frac{\partial L_m}{\partial x_i} = Q_{im} \tag{35}$$

With regard to $\mathrm{d}\boldsymbol{q}_m/\mathrm{d}\alpha_m$, vector $\boldsymbol{q}_m$ is a zero vector except for the entries corresponding to the degrees of freedom connecting the element. Without loss of generality we will assume that they are numbered from 1–4 as in Fig. 2. Vector $\boldsymbol{q}_m$ is

$$\boldsymbol{q}_m^T = \{\cos\alpha_m \quad \sin\alpha_m \quad -\cos\alpha_m \quad -\sin\alpha_m \quad 0 \quad 0 \quad \ldots\} \tag{36}$$

and its derivative with respect to α_m, denoted $\boldsymbol{q}_m'$ becomes

$$\frac{\partial \boldsymbol{q}_m^T}{\mathrm{d}\alpha_m} \equiv \boldsymbol{q}_m^{\prime T} = \{-\sin\alpha_m \quad \cos\alpha_m \quad \sin\alpha_m \quad -\cos\alpha_m \quad 0 \quad 0 \quad \ldots\} \tag{37}$$

Finally, one can show that $\partial\alpha_m/\partial\boldsymbol{x} = \boldsymbol{q}_m'/L_m$ or

$$\frac{\partial\alpha_m}{\partial x_i} = \frac{q_{im}'}{L_m} \tag{38}$$

Introducing (33), (37), (35) and (38) in (31) and (32) gives the sensitivities

$$\frac{\partial s_m}{\partial x_i} = -s_m \frac{q_{im}}{L_m} \tag{39}$$

$$\frac{\partial \boldsymbol{q}_m}{\partial x_i} = \boldsymbol{q}_m' \frac{q_{im}'}{L_m} \tag{40}$$

Here, q_{im} and q_{im}' are the ith components of the corresponding vectors. In the 3-D case the sensitivity of $\boldsymbol{Q}$ leads to similar expressions. See for instance Levy and Spillers (1995).

6. Numerical examples

The geometry optimization procedures described above were implemented with Matlab. A series of structures, of which we will report two examples, were successfully tested. Both examples treat the design of a same structure subjected to a family of disturbances. They differ in their set of design variables. Interestingly in the first example the first singular value of the disturbance matrix was unique whereas in the second example we had a repeated singular value.

The bridge truss in Fig. 1 has 22 nodes and 51 elements. There are 40 DOF, nine of them CDOF. These are the lateral displacements of the lower chord and are designated by arrows in the figure. The disturbances are external lateral forces acting at the CDOF subject to constraint (3). A column i of the disturbance influence matrix is the vector of the lateral displacements of the lower chord for a single disturbance load at CDOF i. In this case the design aims to reduce the deflection of the bridge for a general combination of lateral loads acting on the bridge. These include for instance a load of magnitude one traveling along the bridge, or three loads of magnitude $1/\sqrt{3}$ acting

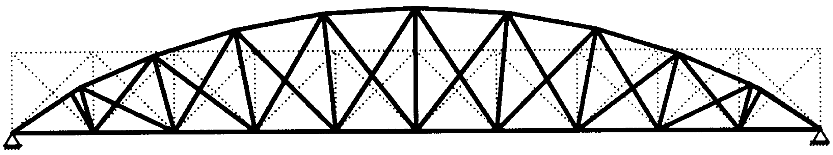

Fig. 3. Optimal geometry of bridge truss without actuators; fixed CDOF.

simultaneously at any one of the DOF or also uniformly distributed loads of magnitude 1/3 applied at all the 9 DOF. The only condition on the external loads is (3). The design variables are the longitudinal and lateral coordinates $\boldsymbol{x}$ of the free nodes and some coordinates of the CDOF. Here two cases are considered: (a) the coordinates of the CDOF are fixed; and (b) the longitudinal coordinates of the CDOF are fixed but the lateral components are part of the design variables. Interestingly a case (c) where the CDOF nodes were free to move in any direction was also checked. This resulted in a trivial solution. The CDOF were lumped at the two supports, resulting indeed in zero deflections at the CDOF!

Considering case (a), the optimized geometry is depicted in Fig. 3. In this case we seek the stiffest geometry having the same mass and connectivity relations as the original structure which will have the least RMS of the CDOF displacements. The optimal geometry is, in accordance with engineering practice, wide in the middle and tapering off towards the supports, resulting in a reduction of about 40% in the first singular value. Thus, for a same amount of material the optimized configuration will be 1.66 stiffer, in the worst distortion case, than the initial design. To have an idea of the optimizer behavior we refer to Fig. 4. This is a graph of the non-dimensional value of σ_1, that is, the value of σ_1 during the optimization divided by its initial value, against a non-dimensional measure of the optimization progress.

We should like to draw attention to this measure since it is rather unusual in structural design, where the abscissa is traditionally the number of iterations. This measure is the cumulative sum of

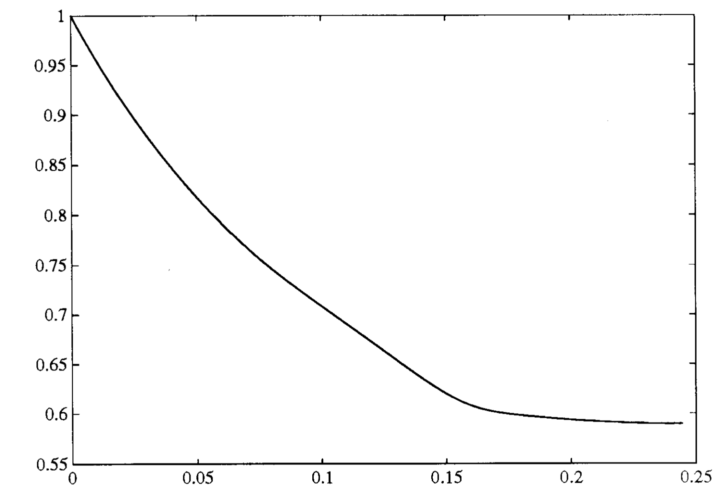

Fig. 4. Bridge truss without control; optimization of first singular value.

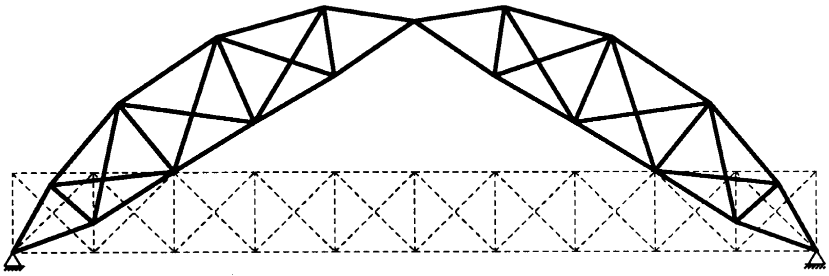

Fig. 5. Optimal geometry of bridge truss without actuators; free CDOF.

the absolute value of the changes in length of the elements, in each step, divided by the total length of the rods. Although the total length (mass) of the structure does not change during the optimization, each element may change in length and therefore the sum of the absolute values is different from zero. This parameter actually tracks the 'flow' of mass in the structure during the optimization.

Returning to Fig. 4 we notice a motonically decreasing curve with large negative derivatives at the beginning and approaching zero slope when closing in on the optimal geometry. In this case the first singular value was and remained distinct.

In case (b) of the same bridge the CDOF nodes were allowed to change their location in the lateral direction. The optimized geometry is shown in Fig. 5. Here we notice a radical conceptual change of the design. Evidently, the optimization algorithm utilized the possibility of modifying the vertical locations of the CDOF nodes and produced a structure which is very different from the original design. The original truss was a beam type while the optimized geometry is a three-hinged structure. The reduction of the first singular value is also drastic, about 85%. The progress of the optimization is shown in Fig. 6 and this time there are repeated singular values. At the start of

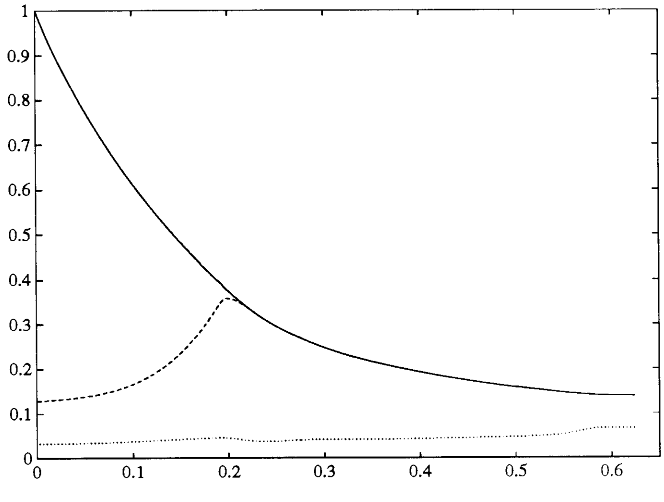

Fig. 6. Optimization of first singular value using 'repeated singular value' algorithm.

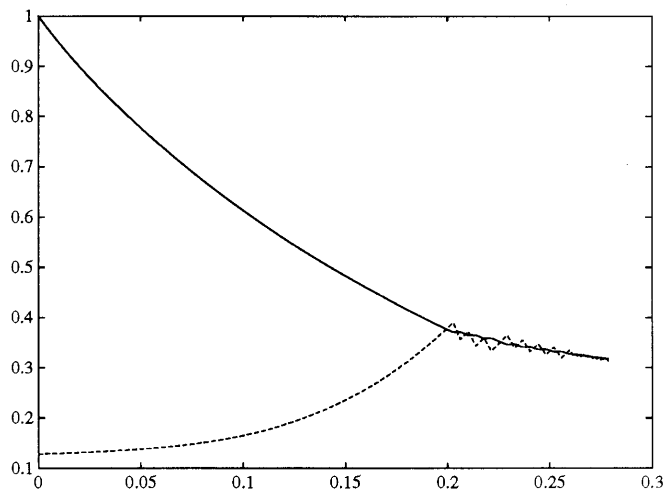

Fig. 7. Optimization of first singular value without using 'repeated singular value' algorithm.

the process the first three singular values of the disturbance matrix are distinct. As the optimization proceeds we note that the second singular value climbs towards the declining first singular value. When they are getting closer the algorithm detects the multiplicity, merges the two singular values and reduces their size simultaneously. The third singular value lingers at the bottom of the graph evidently unaffected by the drastic design changes. Trying to minimize the first singular value without the repeated singular values algorithm resulted in crossovers, as shown in Fig. 7, with a final result well above the optimum. Note, the reconstruction of curves σ_1 and σ_2 involved some guess work. It was not always clear to which curve (σ_1 or σ_2) the maximum value belonged.

7. Conclusions

This paper has presented a method for computing the sensitivities of a repeated singular value of a matrix with respect to parameters affecting the matrix. The derivative for the case of a single singular value can be found in the literature. The derivative for repeated singular values is developed in this paper. The technique is an outgrowth of a similar approach for computing the sensitivities of an eigenvalue of a square matrix. It is shown that the derivatives can be obtained under the condition that the parameter change is in a direction which maintains the order of multiplicity of the singular value.

The method was tested for the design of the geometry of a precision truss subjected to a family of disturbances. In order to maintain the nodal displacements as undeformed as possible one has to minimize the first singular value of the disturbance matrix. This guarantees a minimal value for the worst case distortion under the given family of disturbances. The mathematical programming formulation called for minimizing the first singular value subject to the conditions for maintaining the multiplicity of the singular value in conjunction with a constant volume constraint. This was solved via a Lagrangian multipliers method. The implementation algorithm constantly checked the degree of multiplicity of the singular value and updated that value when necessary.

Two examples were presented: a case with a single singular value and a case where the initially distinct first singular value merges with the climbing second singular value. In the latter example the algorithm detects the incumbent duplicity in due time and one notes the simultaneous reduction of both quantities. An example of the difficulties which arise when not using the repeated singular values algorithm was also given. In this case the crossovers were evident and so was the inability of the algorithm to reach the optimal solution.

Significant reductions of the singular values were achieved, indicating that optimal geometries can indeed enhance the performance of controlled structures.

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